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Mixed correlation functions of the two-matrix model

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Abstract

We compute the correlation functions mixing the powers of two non-commuting random matrices within the same trace. The angular part of the integration was partially known in the literature (Morozov A 1992 *Mod. Phys. Lett.* A **7** 3503, Shatashvili S L 1993 *Commun. Math. Phys.* **154** 421): we pursue the calculation and carry out the eigenvalue integration reducing the problem to the construction of the associated bi-orthogonal polynomials. The generating function of these correlations then becomes a determinant involving the recursion coefficients of the bi-orthogonal polynomials.

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1. Introduction and main result

Random matrix models were first introduced in the context of nuclear physics in order to describe the energy levels statistics for very large nuclei. Wigner proposed the hypothesis that these were distributed as the eigenvalues of a matrix with random entries. Later, random matrix models were used in many areas of physics and mathematics [6, 11, 15].

An important application of random matrices is to two-dimensional gravity, that is, statistical physics on a random surface. In fact, the perturbative expansion of a matrix integral can be accomplished by drawing Feynman graphs on fixed-genus surfaces. Matrix integration can therefore encode the summation over the set of discretized surfaces (possibly carrying some type of matter).

When the parameters of the model are fine tuned near a critical point the average graph's size diverges and macroscopic graphs dominate the sum, so that suitable critical limits can represent statistical models over smooth surfaces. The smooth surfaces that one describes with the aid of matrix models have properties of scale invariance: this means that the critical points of matrix integrals are related to representations of the conformal group. We recall that its finite-dimensional representations are classified by two integers (p, q): it is known that one-matrix models provide instances of $(p, 2)$-irreps only, where as two-matrix models allow to obtain (p, q)-representations.

Possibly the first such application of the two-matrix model was to describe the Ising model on a random surface; in this case the Ising ferromagnetic transition corresponds to the conformal minimal model in the (3, 4)-irrep. To see this, one should associate a color (or spin) + or − to each matrix so that the vertices of one matrix are labelled with a plus sign and the vertices of the other matrix with a minus sign. Then the Feynman graphs generated by a two-matrix model represent discrete surfaces carrying spins (+ and −), i.e. an Ising model on a random surface.

The correlation functions of random matrices generate discrete surfaces with boundaries, and thus are in relationship with boundary conformal theory. A formula for correlation functions representing surfaces with mono-colored boundaries has been known since [10]. It is the aim of this paper to give a formula for a mixed correlation function, i.e. the generating function for discrete surfaces with a bi-colored boundary.

The two-matrix model has attracted a lot of attention recently, and important progress has been made in the study of the associated bi-orthogonal polynomials [3, 5, 13]. In this paper we express the mixed correlation function in terms of the bi-orthogonal polynomials.

1.1. Definition and notation

We consider two $N \times N$ Hermitian matrices M_1, M_2, with a probability measure

$$\begin{aligned} \mathrm{d}\mu(M_1, M_2) &:= \mathcal{Z}_N{}^{-1}\, \mathrm{d}M_1\, \mathrm{d}M_2 \exp[-\mathrm{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)] \\ \mathcal{Z}_N &:= \iint \mathrm{d}M_1\, \mathrm{d}M_2 \exp[-\mathrm{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)] \end{aligned} \tag{1.1}$$

where $\mathrm{d}M_1\,\mathrm{d}M_2$ is the product of Lebesgue measures of all the independent real and imaginary parts of the components of the two matrices divided by the square of the volume of the unitary group $U(N)$ (for later convenience). The functions V_1 and V_2 are called the *potentials* and must be chosen so as to make the integral convergent. The normalization factor $\mathcal{Z}_N$ is called the 'partition function', where the name 'function' refers to its dependence on the two potentials.

One can rewrite the measure in term of eigenvalues and angular integrals [12]:

$$\begin{aligned} \mathrm{d}\mu(M_1, M_2) &:= \mathcal{Z}_N{}^{-1} \Delta(X)^2 \Delta(Y)^2 \\ &\qquad \times \exp\left[-\sum_{i=1}^N (V_1(x_i) + V_2(y_i))\right] \mathrm{e}^{U^\dagger X U V^\dagger Y V} \mathrm{d}U\, \mathrm{d}V \prod_{i=1}^N \mathrm{d}x_i\, \mathrm{d}y_i \\ X &:= \mathrm{diag}(x_1, \ldots, x_N) \qquad Y := \mathrm{diag}(y_1, \ldots, y_N) \qquad U, V \in U(N) \\ \mathcal{Z}_N &:= \iint \prod_{i=1}^N \mathrm{d}x_i\, \mathrm{d}y_i\, \Delta(X)^2 \Delta(Y)^2\, \mathrm{d}U\, \mathrm{d}V \exp\left[-\sum_{i=1}^N (V_1(x_i) + V_2(y_i))\right] \mathrm{e}^{U^\dagger X U V^\dagger Y V} \end{aligned} \tag{1.2}$$

where $\mathrm{d}U\,\mathrm{d}V$ is the product of the *normalized* Haar measure over $U(N) \times U(N)$. In the original two Hermitian matrix model, the integration path for the x_i and y_j is the real axis (location of the eigenvalues of a Hermitian matrix), however, the model can be generalized to include complex paths or their homology classes in the case that the potentials are holomorphic or meromorphic [2, 3].

1.2. Correlation functions

In the applications of the two-matrix model to statistical physics on a random surface, one is interested in computing correlation functions involving traces of products of powers of M_1 and M_2. Each such correlation function can be expanded in Feynman graphs, which represent discrete surfaces with boundaries: the number of boundaries corresponds to the number of traces, the length of each boundary to the total power of M_1 plus the total power of M_2 within the trace [6].

For instance: $\left\langle \mathrm{Tr}\, M_1^r \right\rangle$ is the generating function for discrete surfaces with one boundary (a circle) of length r made of + spins only; $\left\langle \mathrm{Tr}\, M_2^s \right\rangle$ is the generating function for discrete surfaces with one boundary of length s made of − spins only; $\left\langle \mathrm{Tr}\, M_1^r M_2^s \right\rangle$ is the generating function for discrete surfaces with one bi-colored boundary of length $r + s$ made of r + spins, followed by s − spins.

More generally, $\left\langle \mathrm{Tr}\, M_1^{r_1} M_2^{s_1} M_1^{r_2} M_2^{s_2} \dots M_1^{r_n} M_2^{s_n} \right\rangle$ is the generating function for discrete surfaces with one $2n$-colored boundary of length $\sum_i r_i + \sum_i s_i$ made of r_1 + spins, followed by s_1 − spins, followed by r_2 + spins, . . . , followed by s_n − spins.

One may also be interested in 'multi-loop' correlators (i.e., more than one boundary), for instance: $\left\langle \mathrm{Tr}\, M_1^{r_1} \mathrm{Tr}\, M_1^{r_2} \right\rangle_{\mathrm{conn}}$ is the generating function for discrete surfaces with two spin + boundaries, one of length r_1, the other of length r_2. More generally, one may consider correlation functions involving an arbitrary number of traces, each containing arbitrary words of M_1 and M_2.

The correlation functions, with an arbitrary number of traces, with each trace containing powers of only one matrix have been known since the work of [10]. They can be expressed in terms of bi-orthogonal polynomials.

The aim of the present paper is to express also the mixed correlation function

$$\left\langle \mathrm{Tr}\, M_1^r M_2^s \right\rangle \tag{1.3}$$

in terms of bi-orthogonal polynomials and confirming that the key property of these models is that all relevant spectral statistics can be reduced to the computation of the corresponding bi-orthogonal polynomials [4, 10, 14].

1.3. Bi-orthogonal polynomials

Two sequences of monic polynomials

$$\pi_n(x) = x^n + \cdots \qquad \sigma_n(y) = y^n + \cdots \qquad n = 0, 1, \dots \tag{1.4}$$

are called bi-orthogonal if they are 'orthogonal' with respect to a coupled measure on the product space:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{d}y\, \pi_n(x)\sigma_m(y)\, \mathrm{e}^{-V_1(x)-V_2(y)+xy} = h_n\delta_{mn} \qquad h_n \neq 0 \quad \forall n \in \mathbb{N} \tag{1.5}$$

where $V_1(x)$ and $V_2(y)$ are the potentials appearing in the two-matrix model measure (1.1). It is convenient to introduce the associated quasi-polynomial differentials defined by the formulae

$$\psi_n(x) := \frac{1}{\sqrt{h_{n-1}}}\pi_{n-1}(x)\, \mathrm{e}^{-V_1(x)}\, \mathrm{d}x \tag{1.6}$$

$$\phi_n(y) := \frac{1}{\sqrt{h_{n-1}}}\sigma_{n-1}(y)\, \mathrm{e}^{-V_2(y)}\, \mathrm{d}y. \tag{1.7}$$

In terms of these two sequences of differentials the multiplications by x and y, respectively, are represented by semi-infinite square matrices $Q = [Q_{ij}]_{i,j\in\mathbb{N}^*}$ and $P = [P_{ij}]_{i,j\in\mathbb{N}^*}$ according to the formulae

$$\begin{aligned} &x\psi_n(x) = \sum_m Q_{n,m}\psi_m(x) \qquad y\phi_n(y) = \sum_m P_{m,n}\phi_m(y) \\ &Q_{n,m} = 0 = P_{m,n} \qquad \text{if} \quad n > m+1. \end{aligned} \tag{1.8}$$

The matrices P and Q have a rich structure and satisfy the 'string equation' $[P, Q] = \mathrm{i}d$. However, we do not need any of their properties except for equation (1.8) to derive our present results, and therefore we refer for further details to [2–5] where these models are studied especially in the case of polynomial potentials. We also point out that the model can easily be generalized to accommodate contours of integration other than the real axes [2, 3] leaving intact all the properties which are relevant to the following computations.

1.4. The main result

Our goal is to prove a formula for the generating function of the correlators

$$\begin{aligned} \left\langle \mathrm{Tr}\left(M_1{}^r M_2{}^s\right)\right\rangle_{V_1,V_2} &:= \frac{1}{\mathcal{Z}_N}\int \mathrm{d}M_1\,\mathrm{d}M_2\,\mathrm{Tr}\left(M_1{}^r M_2{}^s\right) \\ &\quad\times \exp(-\mathrm{Tr}\,(V_1(M_1) + V_2(M_2) - M_1M_2)). \end{aligned} \tag{1.9}$$

By generating function, we mean the formal double Laurent series

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} := \sum_{r,s} x^{-r-1}y^{-s-1}\left\langle \mathrm{Tr}\left(M_1{}^r M_2{}^s\right)\right\rangle_{V_1,V_2}. \tag{1.10}$$

The main obstacle to this sort of computations so far was posed by the 'angular integration' over the unitary group $U(N)$.

One can trace in the literature various attempts at this computation using the loop equations [7, 8, 18]. A closed formula was found in the large-N limit in [8], but an exact formula for finite-N was never derived.

Our strategy is that of reducing the computation of (1.9) or even better (1.10) to the computation of the corresponding bi-orthogonal polynomials associated with the measure (1.1).

We can now write down the main result of the paper which is both simple and beautiful:

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = 1 - \det\left[\mathbf{1}_N - \underset{N}{\pi}\,\frac{1}{x-Q}\frac{1}{y-P}\,\underset{N}{\pi}^t\right] \tag{1.11}$$

where P and Q are the matrices in equation (1.8) and $\underset{N}{\pi}$ denotes the projector $\mathbb{C}^\infty \mapsto \mathbb{C}^N$ onto the span of the first N canonical basis vectors, i.e., the $N\times\infty$ matrix with non-zero entries $\underset{N}{\pi}_{i,i} = 1, i = 1,\ldots,N$. Formula (1.11) should be properly understood in the sense of an identity of formal Laurent series in the variables x and y, although it would be possible to give an analytic meaning to both sides. For example, from (1.11), one can easily obtain the following identities:

$$\left\langle \mathrm{Tr}\left(M_1{}^r M_2\right)\right\rangle_{V_1,V_2} = \mathrm{Tr}(Q^rP\Pi) - \frac{1}{2}\sum_{j=0}^{r-1}(\mathrm{Tr}(Q^j\Pi)\mathrm{Tr}(Q^{r-1-j}\Pi) - \mathrm{Tr}(Q^j\Pi Q^{r-1-j}\Pi)) \tag{1.12}$$

$$\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^2\right)\right\rangle_{V_1,V_2} = \mathrm{Tr}(Q^r P^2 \Pi) - \sum_{k_1+k_2=r-1} (\mathrm{Tr}(Q^{k_1} P \Pi)\mathrm{Tr}(Q^{k_2}\Pi) - \mathrm{Tr}(Q^{k_1} P \Pi Q^{k_2} \Pi))$$
$$+ \sum_{k_1+k_2+k_3=r-2} \left(\frac{1}{3} \mathrm{Tr}(\Pi Q^{k_1} \Pi Q^{k_2} \Pi Q^{k_3}) - \frac{1}{2} \mathrm{Tr}(\Pi Q^{k_1} \Pi Q^{k_2}) \mathrm{Tr}(\Pi Q^{k_3}) + \frac{1}{6} \mathrm{Tr}(\Pi Q^{k_1}) \mathrm{Tr}(\Pi Q^{k_2}) \mathrm{Tr}(\Pi Q^{k_3}) \right) \tag{1.13}$$

where now $\Pi = \pi^t \pi$ is the semi-infinite square matrix with non-zero entries $\Pi_{i,i} = 1, i = 1, \ldots, N$.

In fact in the course of the proof of equation (1.11) we will also prove the following strong result for the correlations of $\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^s\right)\right\rangle_{V_1,V_2}$

$$\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^r\right)\right\rangle_{V_1,V_2} = \sum_{n=0}^{\min(r,s)} \frac{(-1)^n}{(n+1)!} \int \cdots \int \mathrm{e}^{\sum_k x_k y_k}$$
$$\times \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n+1}^{n-1} \\ x_1^r & \cdots & x_{n+1}^r \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & & \vdots \\ x_1^{n} & \cdots & x_{n+1}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_{n+1} \\ \vdots & & \vdots \\ y_1^{n-1} & \cdots & y_{n+1}^{n-1} \\ y_1^s & \cdots & y_{n+1}^s \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_{n+1} \\ \vdots & & \vdots \\ y_1^{n} & \cdots & y_{n+1}^{n} \end{pmatrix}}$$
$$\times \det[K_{12}(x_k, y_\ell)]_{k,\ell \leqslant n+1}. \tag{1.14}$$

The formula can be extended to arbitrary analytic functions $f(x)$ and $g(y)$ to give

$$\langle \mathrm{Tr}(f(M_1) g(M_2))\rangle_{V_1,V_2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \int \cdots \int \mathrm{e}^{\sum_k x_k y_k}$$
$$\times \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n+1}^{n-1} \\ f(x_1) & \cdots & f(x_{n+1}) \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & & \vdots \\ x_1^{n} & \cdots & x_{n+1}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_{n+1} \\ \vdots & & \vdots \\ y_1^{n-1} & \cdots & y_{n+1}^{n-1} \\ g(y_1) & \cdots & g(y_{n+1}) \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_{n+1} \\ \vdots & & \vdots \\ y_1^{n} & \cdots & y_{n+1}^{n} \end{pmatrix}}$$
$$\times \det[K_{12}(x_k, y_\ell)]_{k,\ell \leqslant n+1}. \tag{1.15}$$

Here we have used the kernel constructed from the quasi-polynomials [10],

$$K_{12}(x,y) := \sum_{n=1}^{N} \psi_n(x)\phi_n(y) = \sum_{n=0}^{N-1} \frac{\pi_n(x)\sigma_n(y)\,\mathrm{e}^{-V_1(x)-V_2(y)}}{h_n}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.16}$$

2. Proofs of formulae (1.11) and (1.14)

We want to compute the expectation values

$$\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^s\right)\right\rangle_{V_1,V_2} := \frac{1}{\mathcal{Z}_N}\int \mathrm{d}\mu(M_1,M_2)\,\mathrm{Tr}\left({M_1}^r {M_2}^s\right). \tag{2.1}$$

We denote by $\{x_i\}_{i=1,\dots,N}$ and $\{y_i\}_{i=1,\dots,N}$ the spectra of the two matrices M_1, M_2 and by $U \in U(N)$ the relative angles. Then we can reduce the integral (2.1) to an integral over the spectra of M_1, M_2 and the unitary group of the relative angles. Indeed we have

$$\begin{aligned}\left\langle \mathrm{Tr}\left(M_1^r M_2^s\right)\right\rangle_{V_1,V_2} &:= \frac{1}{\mathcal{Z}_N}\int \mathrm{d}\mu(M_1,M_2)\,\mathrm{Tr}\left(M_1^r M_2^s\right)\\ &= \frac{1}{\mathcal{Z}_N}\int \prod_{i=1}^{N} \mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{e}^{-V_1(x_i)-V_2(y_i)}\,\Delta^2(X)\,\Delta^2(Y)\\ &\quad\times \sum_{i,j=1}^{N} {x_i}^r {y_j}^s \int_{U(N)} \mathrm{d}U\,|U_{ji}|^2\,\mathrm{e}^{\mathrm{Tr}(XU^\dagger YU)}\end{aligned} \tag{2.2}$$

where $X = \mathrm{diag}(x_1,\dots,x_N), Y = \mathrm{diag}(y_1,\dots,y_N)$ and $\Delta(X), \Delta(Y)$ denote the Vandermonde determinants. The computation requires the knowledge of the two-point correlator for the unitary integral in equation (2.2) which we analyse in the next subsection.

2.1. Two-point correlator for the unitary integral

The computation of this quantity has been considered in the finite-N regime in the two papers [16, 17]. In [17] was described a complete algorithm that allows the construction of a formula for the most general correlator

$$\left\langle U_{i_1j_1}U^\dagger_{k_1l_1}\cdots U_{i_nj_n}U^\dagger_{k_nl_n}\right\rangle_{U(N)} := \int_{U(N)} \mathrm{d}U\,U_{i_1j_1}\,U^\dagger_{k_1l_1}\cdots U_{i_nj_n}\,U^\dagger_{k_nl_n}\,\mathrm{e}^{\mathrm{Tr}(XU^\dagger YU)}. \tag{2.3}$$

Such an algorithm involves the introduction of the parametrization of the unitary group given by the Gel'fand–Tsetlin coordinates associated with one of the two matrices M_i; however, the computation is highly involved and in [17] the problem was not carried through to a completely manageable formula.

On the other hand in [16], a very simple closed formula was proposed for the two-point correlator $\langle |U_{ji}|^2\rangle_{U(N)}$ in terms of a generating function, that is

$$\begin{aligned}\sum_{i,j} a_i b_j \langle |U_{ji}|^2\rangle_{U(N)} &= \sum_{i,j=1}^{N} a_i b_j \int_{U(N)} \mathrm{d}U\,|U_{ji}|^2\,\mathrm{e}^{\mathrm{Tr}(XU^\dagger YU)}\\ &= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho)\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}} \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}}\end{aligned}$$

$$\times \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ a_{i_1} & \cdots & a_{i_{n+1}} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n-1} & \cdots & y_{\rho(i_{n+1})}^{n-1} \\ b_{\rho(i_1)} & \cdots & b_{\rho(i_{n+1})} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n} & \cdots & y_{\rho(i_{n+1})}^{n} \end{pmatrix}} \tag{2.4}$$

with the understanding that the Haar measure of the unitary group $U(N)$ has been normalized to unity. This formula will be the starting point of our analysis: however, the author of [16] did not actually prove the formula but just made an educated (and—as it turns out—correct) guess.

Therefore, before proceeding to proving our main result (1.11) we want to fill in the gaps between the full but unpractical algorithm given in [17] and the practical but unproven formula in [16]. We do not need the full generality of [17]: our departure point is formula (1.4) of [17], restricted to the particular case of the two-point correlator. For the ease of the reader we rewrite the aforementioned formula in the notation of our present paper

$$\begin{aligned} \left\langle U_{1j_1}U^\dagger_{k_1 1}\cdots U_{1j_n}U^\dagger_{k_n 1}\right\rangle_{U(N)} &= \frac{\delta_{j_1k_1}\cdots\delta_{j_nk_n}}{\Delta(X)\Delta(y_2,\ldots,y_N)}\left(\prod_{k=1}^{N-1}\int_{x_k}^{x_{k+1}}\mathrm{d}\xi_k\right) \\ &\quad\times\frac{\prod_{\ell=1}^{N-1}(\xi_\ell-x_{j_1})\cdots\prod_{\ell=1}^{N-1}(\xi_\ell-x_{j_n})}{\prod_{\ell\neq j_1}(x_\ell-x_{j_1})\cdots\prod_{\ell\neq j_n}(x_\ell-x_{j_n})} \\ &\quad\times\exp\left[\sum_{k=1}^{N}x_ky_1-\sum_{k=1}^{N-1}\xi_ky_1\right]\det[\mathrm{e}^{\xi_iy_{j+1}}]_{i,j=1,\ldots,N-1}. \end{aligned} \tag{2.5}$$

Here the Haar measure of the unitary group has been normalized to unity, $\Delta(y_2,\ldots,y_N)$ denotes the Vandermonde determinant of the $N-1$ numbers $y_2,\ldots,y_N$, while $\Delta(X)$ is the short form for the Vandermonde determinant of the whole spectrum of the matrix X.

Equation (2.5) was proved rigorously in [17] for $x_1 < x_2 < \cdots < x_N$; however, it is straightforward to realize that the unitary integral in equation (2.2) defines an analytic (in fact entire) function of the variables X and Y. Therefore, the result extends to statistical ensembles of pairs M_1, M_2 of *normal matrices*[4] with their spectrum on arbitrary paths on the complex plane. Moreover, the restriction on the order of the spectrum can be lifted because the result is analytic in the variables $x_1,\ldots,x_N$ and can be analytically continued to $\mathbb{C}^N$.

[4] We recall that a normal matrix is a matrix that commutes with its Hermitian transposed. Any such matrix can be diagonalized using a unitary transformation (and vice versa).

We now set $n = 1$ in equation (2.5) and hence $k_1 = j_1 = i$. Then we have

$$\langle|U_{1i}|^2\rangle_{U(N)} = \frac{\mathrm{e}^{\sum^N x_\ell y_1}}{\Delta(X)\Delta(y_2,\ldots,y_N)\prod_{\ell\neq i}(x_\ell - x_i)}$$
$$\times \sum_{\rho\in\mathcal{S}_{N-1}} \epsilon(\rho) \prod_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} \mathrm{d}\xi_k (\xi_k - x_i)\, \mathrm{e}^{\xi_k(y_{\rho(k)+1}-y_1)}$$
$$= \frac{\mathrm{e}^{\sum^N x_\ell y_1}}{\Delta(X)\Delta(y_2,\ldots,y_N)\prod_{\ell\neq i}(x_\ell - x_i)} \sum_{\rho\in\mathcal{S}_{N-1}} \epsilon(\rho)$$
$$\left[\left(\frac{x_{k+1}-x_i}{y_{\rho(k)+1}-y_1} - \frac{1}{(y_{\rho(k)+1}-y_1)^2}\right)\mathrm{e}^{x_{k+1}(y_{\rho(k)+1}-y_1)}\right.$$
$$\left. - \left(\frac{x_k - x_i}{y_{\rho(k)+1}-y_1} - \frac{1}{(y_{\rho(k)+1}-y_1)^2}\right)\mathrm{e}^{x_k(y_{\rho(k)+1}-y_1)}\right]$$
$$= \frac{\mathrm{e}^{\sum^N x_\ell y_1}}{\Delta(X)\Delta(Y)\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq 1}(y_\ell - y_1)} \sum_{\rho\in\mathcal{S}_{N-1}} \epsilon(\rho)$$
$$[((x_{k+1}-x_i)(y_{\rho(k)+1}-y_1)-1)\,\mathrm{e}^{x_{k+1}(y_{\rho(k)+1}-y_1)} - ((x_k - x_i)(y_{\rho(k)+1}-y_1)-1)\,\mathrm{e}^{x_k(y_{\rho(k)+1}-y_1)}]. \tag{2.6}$$

We observe that the following identity holds

$$\mathrm{e}^{\sum^N x_\ell y_1} \sum_{\rho\in\mathcal{S}_{N-1}} \epsilon(\rho)[((x_{k+1}-x_i)(y_{\rho(k)+1}-y_1)-1)\,\mathrm{e}^{x_{k+1}(y_{\rho(k)+1}-y_1)}$$
$$- ((x_k - x_i)(y_{\rho(k)+1}-y_1)-1)\,\mathrm{e}^{x_k(y_{\rho(k)+1}-y_1)}]$$
$$= -\det[((x_\ell - x_i)(y_m - y_1)-1)\,\mathrm{e}^{x_\ell y_m}]_{\ell,m=1,\ldots,N} \tag{2.7}$$

which is realized by performing elementary row operations on the matrix inside the determinant in (2.7) so that the $N \times N$ determinant reduces to a $(N-1)^2$ determinant by use of Laplace's formula.

The more general case of the expectation value of the (i, j) element is obtained by permutation of the spectrum of the matrix Y so as to give the formula

$$\langle|U_{ji}|^2\rangle_{U(N)} = \frac{-1}{\Delta(X)\Delta(Y)} \frac{\det[((x_\ell - x_i)(y_m - y_j)-1)\mathrm{e}^{x_\ell y_m}]_{\ell,m=1,\ldots,N}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}. \tag{2.8}$$

This form of Shatashvili's formula (2.5) for the case $n = 1$ is remarkably simple but not suitable for our later purposes. Moreover, it is not yet clearly equivalent to Morozov's formula (2.4), which is what we need for our computation. It will be proved in the appendix that the two formulae are indeed equivalent.

2.2. *The correlators* $\left\langle \mathrm{Tr}\, {M_1}^r {M_2}^s \right\rangle_{V_1,V_2}$ *and their generating function*

Starting from equation (2.2) and using formula (2.4) with $a_i = {x_i}^r$ and $b_j = {y_j}^s$ we obtain

$$\int_{U(N)} \mathrm{d}U\, \mathrm{Tr}(X^r U^\dagger Y^s U)\, \mathrm{e}^{\mathrm{Tr}(XU^\dagger YU)} = \sum_{i,j=1}^{N} x_i{}^r y_j{}^s \int_{U(N)} \mathrm{d}U |U_{ji}|^2\, \mathrm{e}^{\mathrm{Tr}(XU^\dagger YU)}$$

$$= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho)\, \mathrm{e}^{\sum x_\ell y_{\rho(\ell)}} \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}}$$

$$\times \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ x_{i_1}^{r} & \cdots & x_{i_{n+1}}^{r} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n-1} & \cdots & y_{\rho(i_{n+1})}^{n-1} \\ y_{\rho(i_1)}^{s} & \cdots & y_{\rho(i_{n+1})}^{s} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n} & \cdots & y_{\rho(i_{n+1})}^{n} \end{pmatrix}} \tag{2.9}$$

with the understanding that we use the normalized Haar measure over the unitary group $U(N)$ and that for $n = 0$ the ratio of determinants should be $x_{i_1}^r y_{\rho(i_1)}^s$. The first observation is that the sum over n does not actually need to be extended up to the size N of the random matrices because the determinants will vanish for $n > \min(r, s)$. The next remark is that the ratios of determinants actually define certain totally symmetric polynomials of their arguments of degree $r - n$ and $s - n$, respectively: in fact they are Schur polynomials corresponding to hook Young diagrams

$$S_r(x_{i_1}, \ldots, x_{i_{n+1}}) := \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ x_{i_1}^{r} & \cdots & x_{i_{n+1}}^{r} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}}$$

$$= \sum_{a_1\leqslant a_2\leqslant\cdots\leqslant a_{r-n}} \prod_{k=1}^{r-n} x_{i_{a_k}} = \sum_{j_1+\cdots+j_{n+1}=r-n} x_{i_1}^{j_1} \cdots x_{i_{n+1}}^{j_{n+1}} \tag{2.10}$$

and a similar expression for the y part. It is interesting to note that in equation (2.9) the characters of the representations of the group $GL(N)$ appear; the same equation could possibly be derived from the character expansion of the integrand.

The formal generating function of these Schur polynomials is

$$\sum_{r=0}^{\infty} \frac{1}{x^{r+1}} S_r\left(x_{i_1}, \ldots, x_{i_{n+1}}\right) = \prod_{k=1}^{n+1} \frac{1}{x - x_{i_k}}. \tag{2.11}$$

Equation (2.2) now becomes

$$\int \mathrm{d}\mu(M_1, M_2)\,\mathrm{Tr}\big(M_1^r M_2^s\big) = \int \prod_{i=1}^{N} \mathrm{d}\mu(x_i)\mathrm{d}\nu(y_i)\,\Delta(X)\,\Delta(Y) \sum_{\rho\in S_N} \epsilon(\rho)\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}$$
$$\times \sum_{n=0}^{\min(r,s)} (-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}} S_r\big(x_{i_1}, \ldots, x_{i_{n+1}}\big) S_s\big(y_{\rho(i_1)}, \ldots, y_{\rho(i_{n+1})}\big) \tag{2.12}$$

where $\mathrm{d}\mu(x) := \mathrm{e}^{-V_1(x)}\,\mathrm{d}x$ and $\mathrm{d}\nu(y) := \mathrm{e}^{-V_2(y)}\,\mathrm{d}y$.

Using equation (2.11) we have

$$\int \mathrm{d}\mu(M_1, M_2)\,\mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)$$
$$= \int \prod_{i=1}^{N} \mathrm{d}\mu(x_i)\,\mathrm{d}\nu(y_i)\,\Delta(X)\,\Delta(Y) \sum_{\rho\in S_N} \epsilon(\rho)\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}$$
$$\times \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}} \prod_{k=1}^{n+1} \frac{1}{\big(x-x_{i_k}\big)\big(y-y_{\rho(i_k)}\big)}. \tag{2.13}$$

By a relabelling of the y the sum over ρ becomes an overcounting factor $N!$:

$$\int \mathrm{d}\mu(M_1, M_2)\,\mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right) = N! \int \prod_{i=1}^{N} \mathrm{d}\mu(x_i)\,\mathrm{d}\nu(y_i)\,\Delta(X)\,\Delta(Y)\,\mathrm{e}^{\sum x_\ell y_\ell}$$
$$\times \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}} \prod_{k=1}^{n+1} \frac{1}{\big(x-x_{i_k}\big)\big(y-y_{i_k}\big)}. \tag{2.14}$$

This formula allows us to obtain the following expression for the expectations

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2}$$
$$= \frac{N!}{\mathcal{Z}_N} \int \prod_{i=1}^{N} \mathrm{d}\mu(x_i)\,\mathrm{d}\nu(y_i)\,\Delta(X)\,\Delta(Y)\,\mathrm{e}^{\sum x_\ell y_\ell}$$
$$\times \sum_{n=0}^{N-1} (-1)^n \sum_{i_1<\cdots<i_{n+1}} \prod_{k=1}^{n+1} \frac{1}{\big(x-x_{i_k}\big)\big(y-y_{i_k}\big)}$$
$$= \frac{1}{N!} \sum_{n=0}^{N-1} (-1)^n \sum_{\sigma,\tau\in S_N} \epsilon(\sigma\tau) \sum_{i_1<i_2<\cdots<i_{n+1}} \int \prod_{j=1}^{N} \psi_{\sigma(j)}(x_j)\phi_{\tau(j)}(y_j)\,\mathrm{e}^{x_j y_j}$$
$$\times \prod_{k=1}^{n+1} \frac{1}{\big(x-x_{i_k}\big)\big(y-y_{i_k}\big)}. \tag{2.15}$$

In equation (2.15) we have used the normalized quasi-polynomial differentials defined in (1.7), the fact that, with our normalizations for the Haar measure of $U(N)$, the partition function is $\mathcal{Z}_N = (N!)^2 \prod_{j=0}^{N-1} h_j$ (see [14]), and the identities

$$\Delta(X) \prod_{k=1}^{N} \mathrm{d}\mu(x_k) = \left(\prod_{j=0}^{N-1} \sqrt{h_j}\right) \sum_{\sigma\in S_N} \epsilon(\sigma) \prod_{k=1}^{N} \psi_{\sigma(k)}(x_k) \tag{2.16}$$

$$\Delta(Y)\prod_{k=1}^{N}\mathrm{d}\nu(y_k) = \left(\prod_{j=0}^{N-1}\sqrt{h_j}\right)\sum_{\tau\in S_N}\epsilon(\tau)\prod_{k=1}^{N}\phi_{\tau(k)}(y_k) \tag{2.17}$$

which are obtained by replacing the monomials in the Vandermonde determinants by the bi-orthogonal polynomials of the same degree.

2.2.1. Proof of formula (1.14). We are now in the position of proving formula (1.14) in a few strokes. Taking the coefficient of $x^r y^s$ from the formal generating function in equation (2.15) we obtain the expression

$$\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^s\right)\right\rangle_{V_1,V_2} = \frac{1}{N!}\sum_{n=0}^{\min(r,s)}(-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}}\sum_{\sigma,\tau\in S_N}\epsilon(\sigma\tau)$$

$$\times\int\prod_{j=1}^{N}\psi_{\sigma(j)}(x_j)\phi_{\tau(j)}(y_j)\,\mathrm{e}^{x_j y_j}\,S_r\left(x_{[i]_{n+1}}\right)S_s\left(y_{[i]_{n+1}}\right) \tag{2.18}$$

$$= \frac{1}{N!}\sum_{n=0}^{\min(r,s)}(-1)^n \sum_{i_1<i_2<\cdots<i_{n+1}}\sum_{\sigma,\tau\in S_N}\epsilon(\sigma\tau)$$

$$\times\int\prod_{j\notin\{i_1,\ldots,i_{n+1}\}}^{N}\psi_{\sigma(j)}(x_j)\phi_{\tau(j)}(y_j)\,\mathrm{e}^{x_j y_j} \tag{2.19}$$

$$\times\int\prod_{k=1}^{n+1}\psi_{\sigma(i_k)}\left(x_{i_k}\right)\phi_{\tau(i_k)}\left(y_{i_k}\right)\mathrm{e}^{x_{i_k}y_{i_k}}\,S_r\left(x_{[i]_{n+1}}\right)\,S_s\left(y_{[i]_{n+1}}\right). \tag{2.20}$$

In this formula the notation $x_{[i]_{n+1}}$ means the sequence of variables $x_{i_1},\ldots,x_{i_{n+1}}$ (similarly for the y) and we have used the fact that the Schur polynomials S_r as defined in (2.10) vanish if the number of variables is greater than r. Next, the orthogonality relations between the ϕ_ns and the ψ_ns in line (2.19) imply that the sum over the permutations σ, τ is restricted to those permutations such that

$$\sigma = \tau\circ\eta \qquad \eta\in S\{i_1,\ldots,i_{n+1}\} \tag{2.21}$$

where $S\{i_1,\ldots,i_{n+1}\}$ denotes the group of permutation of the indices i_k. The restriction on the indices $i_1,\ldots,i_{n+1}$ can be lifted because the following expression is permutation invariant in the label of those indices and when two such indices coincide the corresponding term vanishes due to the alternating form of the sum. This will produce an overcounting of a factor $(n+1)!$ which must be corrected: moreover, we can relabel the variables of integration of the integral in line (2.20) from x_{i_k}, y_{i_k} to x_k, y_k

$$\left\langle \mathrm{Tr}\left({M_1}^r {M_2}^s\right)\right\rangle_{V_1,V_2} = \frac{1}{N!}\sum_{\sigma\in S_N}\sum_{n=0}^{\min(r,s)}\frac{(-1)^n}{(n+1)!}\sum_{i_1,i_2,\ldots,i_{n+1}}\sum_{\eta\in S_{n+1}}\epsilon(\eta)$$

$$\times\int \mathrm{e}^{\sum_k^{n+1}x_k y_k}\,S_r(x_{[1,\ldots]})S_s(y_{[1,\ldots]})\prod_{k=1}^{n+1}\psi_{i_k}(x_k)\phi_{i_{\eta(k)}}(y_k)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\min(r,s)} \frac{(-1)^n}{(n+1)!} \sum_{i_1,i_2,\ldots,i_{n+1}} \sum_{\eta\in S_{n+1}} \epsilon(\eta) \\
&\quad \times \int \mathrm{e}^{\sum_k^{n+1} x_k y_k} S_r(x_{[1,\ldots,n+1]})\, S_s(y_{[1,\ldots,n+1]}) \prod_{k=1}^{n+1} \psi_{i_k}(x_k)\phi_{i_k}(y_{\eta^{-1}(k)}) \\
&= \sum_{n=0}^{\min(r,s)} \frac{(-1)^n}{(n+1)!} \sum_{i_1,i_2,\ldots,i_{n+1}} \sum_{\eta\in S_{n+1}} \epsilon(\eta) \\
&\quad \times \int \mathrm{e}^{\sum_k^{n+1} x_k y_k} S_r(x_{[1,\ldots,n+1]}) S_s(y_{[1,\ldots,n+1]}) \prod_{k=1}^{n+1} \psi_{i_k}(x_k)\phi_{i_{\eta(k)}}(y_k) \\
&= \sum_{n=0}^{\min(r,s)} \frac{(-1)^n}{(n+1)!} \sum_{i_1,i_2,\ldots,i_{n+1}} \sum_{\eta\in S_{n+1}} \epsilon(\eta) \\
&\quad \times \int \mathrm{e}^{\sum_k^{n+1} x_k y_k} S_r(x_{[1,\ldots,n+1]})\, S_s(y_{[1,\ldots,n+1]}) \prod_{k=1}^{n+1} \psi_{i_k}(x_k)\phi_{i_k}(y_{\eta^{-1}(k)}) \\
&= \sum_{n=0}^{\min(r,s)} \frac{(-1)^n}{(n+1)!} \int \mathrm{e}^{\sum_k^{n+1} x_k y_k} S_r(x_{[1,\ldots,n+1]})\, S_s(y_{[1,\ldots,n+1]}) \det[K_{12}(x_j, y_\ell)]_{j,\ell\leqslant n+1}
\end{aligned}
\tag{2.22}
$$

where we have used the definition of the kernel $K_{12}(x, y)$ given in equation (1.16). This concludes the proof of equation (1.14) from which formula (1.15) follows immediately.

2.2.2. Proof of formula (1.11). Resuming from equation (2.15) and performing a relabelling of the x and the y allows to choose $i_k = k$, and the sum over $i_1 < \cdots < i_{n+1}$ becomes a combinatorial factor:

$$
\begin{aligned}
\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} &= \frac{1}{N!}\sum_{n=0}^{\infty}(-1)^n \binom{N}{n+1} \sum_{\sigma,\tau\in S_N} \epsilon(\sigma\tau) \\
&\quad \times \int \prod_{j=1}^{N} \psi_{\sigma(j)}(x_j)\phi_{\tau(j)}(y_j)\, \mathrm{e}^{x_j y_j} \prod_{k=1}^{n+1} \frac{1}{(x-x_k)(y-y_k)}.
\end{aligned}
\tag{2.23}
$$

Now, equations (1.5) and (1.8) imply that

$$
\int \psi_{\sigma(j)}(x')\phi_{\tau(j)}(y')\,\mathrm{e}^{x'y'} = \delta_{\sigma(j),\tau(j)} \tag{2.24}
$$

and

$$
\int \psi_{\sigma(j)}(x')\phi_{\tau(j)}(y')\,\mathrm{e}^{x'y'} \frac{1}{(x-x')(y-y')} = W_{\sigma(j),\tau(j)} \tag{2.25}
$$

where W is the $N \times N$ square matrix[5]:

$$
W := \underset{N}{\pi} \frac{1}{x-Q}\frac{1}{y-P} \underset{N}{\pi^t}. \tag{2.26}
$$

[5] We remind the reader that this matrix should be properly understood as a formal power series in inverse powers of x and y, although an analytic definition could be given in terms of the bi-orthogonal polynomials. However, this is unnecessary for the scope of the present paper.

Therefore $\tau = \sigma\eta$ where η is a permutation of the $n+1$ first indices only, and we can write

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = \frac{1}{N!}\sum_{n=0}^{\infty}(-1)^n \binom{N}{n+1}\sum_{\sigma\in S_N}\sum_{\eta\in S_{n+1}}\epsilon(\eta)\prod_{j=1}^{n+1}W_{\sigma(j),\sigma\eta(j)}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!(N-n-1)!}\sum_{\sigma\in S_N}\det(W_{\sigma(i),\sigma(j)})_{i,j=1,\ldots,n+1}. \tag{2.27}$$

We note $\alpha_i = \sigma(i)$ and we write

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!(N-n-1)!}\sum_{\alpha_1\neq\cdots\neq\alpha_N}\det\left(W_{\alpha_i,\alpha_j}\right)_{i,j=1,\ldots,n+1} \tag{2.28}$$

where the sum over $\alpha_{n+2},\ldots,\alpha_N$ disappears and brings a factor $(N-n-1)!$:

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\sum_{\alpha_1\neq\cdots\neq\alpha_{n+1}}\det\left(W_{\alpha_i,\alpha_j}\right)_{i,j=1,\ldots,n+1}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\sum_{\alpha_1\neq\cdots\neq\alpha_{n+1}}\sum_{\eta\in S_{n+1}}\epsilon(\eta)\prod_{j=1}^{n+1}W_{\alpha_i,\alpha_{\eta(j)}} \tag{2.29}$$

and we can replace the sum over distinct α by a sum over all α:

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\sum_{\eta\in S_{n+1}}\epsilon(\eta)\sum_{\alpha_1,\ldots,\alpha_{n+1}}\prod_{j=1}^{n+1}W_{\alpha_i,\alpha_{\eta(j)}}. \tag{2.30}$$

If η is decomposed into a product of $\mathcal{N}(\eta)$ cyclic permutations of lengths $l_1(\eta)+\cdots+l_{\mathcal{N}(\eta)}(\eta) = n+1$, we have

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\sum_{\eta\in S_{n+1}}\prod_{j=1}^{\mathcal{N}(\eta)}(-1)^{l_j(\eta)+1}\,\mathrm{Tr}\,W^{l_j(\eta)}$$
$$= -\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{\eta\in S_{n+1}}\prod_{j=1}^{\mathcal{N}(\eta)}(-\mathrm{Tr}\,W^{l_j(\eta)}). \tag{2.31}$$

Now we use the following lemma, which is a classical result in combinatorics:

Lemma 2.1. *Let $\mathcal{G}_m$ be a function defined on the permutation group of m elements S_m with the* cluster *property, i.e., such that if $\eta = \eta_1 \circ \eta_2$ is a decomposition into disjoint permutations of m' and m'' elements (and hence $m = m' + m''$) then*

$$\mathcal{G}_m(\eta) = \mathcal{G}_{m'}(\eta_1)\mathcal{G}_{m''}(\eta_2). \tag{2.32}$$

Under these circumstances we have the identity

$$\exp\left(\sum_{m=1}^{\infty}\sum_{\sigma\in\mathcal{C}_m}\frac{x^m}{m!}\mathcal{G}_m(\sigma)\right) = 1 + \sum_{m=1}^{\infty}\frac{x^m}{m!}\sum_{\eta\in S_m}\mathcal{G}_m(\eta) \tag{2.33}$$

where $\mathcal{C}_m$ denotes the set of all permutations of maximal length and has cardinality $(m-1)!$.

In other words, this lemma says that, if $\mathcal{G}$ has the cluster property, then taking the logarithm of the RHS of equation (2.33) removes all 'nonconnected' contributions and returns the 'connected components' only. In view of lemma 2.1 let us define

$$\mathcal{G}_m(\eta) := \prod_{j=1}^{\mathcal{N}(\eta)}(-\mathrm{Tr}\, W^{l_j(\eta)}) \tag{2.34}$$

which has clearly the cluster property, and we have

$$\begin{aligned}1 - \left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{1}{y-M_2}\right)\right\rangle_{V_1,V_2} &= 1 + \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\eta\in S_m}\mathcal{G}_m(\eta) = \exp\left[\sum_{m=1}^{\infty}\sum_{\eta\in\mathcal{C}_m}\frac{1}{m!}\mathcal{G}_m(\eta)\right]\\ &= \exp\left[-\sum_{m=1}^{\infty}\frac{\mathrm{Tr}\, W^m}{m}\right] = \exp[\mathrm{Tr}\ln(\mathbf{1}_N - W)] = \det(\mathbf{1}_N - W).\end{aligned} \tag{2.35}$$

This concludes our proof of formula (1.11). □

3. Conclusions

Formula (1.11) is quite simple in spite of the long computations involved in its proof. It is tempting to imagine that more complicated multi-correlators could also be reduced to a computation involving the matrices P and Q, i.e., to bi-orthogonal polynomials. Quite clearly the possibility rests on having a manageable formula for the generic multi-correlators of the Itzykson–Zuber–Harish-Chandra integral. There are indications that such a formula should be derivable: for example, it is quite simple to obtain a formula for a correlator of entries with the same first index. Indeed starting from equation (2.5) and proceeding in the same way we did in order to obtain equation (2.8) one can easily prove the following formula

$$\left\langle\prod_{a=1}^{n}\left|U_{ji_a}\right|^2\right\rangle_{U(N)} = \frac{(-1)^n}{n!}\frac{\det[F_m(x_\ell)\,\mathrm{e}^{x_\ell y_m}]_{\ell,m=1,\ldots,N}}{\Delta(X)\Delta(Y)\prod_{a=1}^{n}\prod_{\ell\neq i_a}\left(x_\ell - x_{i_a}\right)\prod_{\ell\neq j}(y_\ell - y_j)^n} \tag{3.1}$$

where

$$\begin{aligned}F_m(\xi) &:= \left[(y_m-y_j)^n\prod_{a=1}^{n}\left(\xi - x_{i_a}\right) - (y_m-y_j)^{n-1}\sum_{\ell=1}^{n}\prod_{a\neq\ell}^{n}\left(\xi - x_{i_a}\right) + \cdots + (-1)^n n!\right]\\ &= \sum_{s=0}^{n}(-1)^s(y_m-y_j)^{n-s}\frac{\mathrm{d}^s}{\mathrm{d}\xi^s}\prod_{a=1}^{n}\left(\xi - x_{i_a}\right).\end{aligned} \tag{3.2}$$

However, the computation for non-equal first indices quickly becomes extremely complicated at least using the technique in [17]. Nonetheless we hope that this first computation can break through the general belief that computations of correlators in multi-matrix models are not feasible in the finite-N regime due to the angular integrations.

It should also be remarked that for polynomial potentials we could use the so-called 'loop' equations to obtain information on other correlators. The result, however, would be dependent on the specific form of the potentials and would not provide information on the HCIZ integral itself.

Let us also mention that this calculation could be generalized to other random matrix models, in particular the complex matrix model, which presents many similarities with the two-matrix model. The Gaussian complex-matrix model has attracted a lot of attention in String theory. A particular case of the ADS/CFT, is the conjectured duality between string theory in a pp-wave background, and BMN gauge theory. The Gaussian complex-matrix model appears as an effective BMN theory in a particular limit, and the computation of mixed correlation functions is very important in that model [1, 9]. In the Gaussian complex-matrix model, a formula is known for the two-point mixed correlator $\left\langle \mathrm{Tr}\, \frac{1}{x-M}\frac{1}{y-M^\dagger}\right\rangle$, but little is known for other mixed correlation functions.

Appendix. Equivalence of equations (2.4) and (2.8)

In this appendix, we prove that the two formulae (2.4) and (2.8) are equivalent. To this end we start from (2.8) and compute

$$
\begin{aligned}
\langle |U_{ji}|^2\rangle_{U(N)} &= \frac{-1}{\Delta(X)\Delta(Y)} \frac{\det[((x_\ell - x_i)(y_m - y_j) - 1)\,\mathrm{e}^{x_\ell y_m}]_{\ell,m=1,\ldots,N}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)} \\
&= \frac{-1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \frac{\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)} \\
&\quad\times \prod_{\ell=1}^{N}((x_\ell - x_i)(y_{\rho(\ell)} - y_j) - 1) \\
&= \frac{-1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \frac{(-1)^N\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)} \\
&\quad\times \prod_{\ell=1}^{N}(1 - (x_\ell - x_i)(y_{\rho(\ell)} - y_j)) \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \frac{(-1)^{N+1}\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}
\end{aligned}
$$

$$
\left[1 - \sum_{\ell=1}^{N}(x_\ell - x_i)(y_{\rho(\ell)} - y_j) + \sum_{\ell_1<\ell_2}\big(x_{\ell_1} - x_i\big)\big(y_{\rho(\ell_1)} - y_j\big)\big(x_{\ell_2} - x_i\big)\big(y_{\rho(\ell_2)} - y_j\big) + \cdots\right]
$$

$$
\begin{aligned}
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \frac{(-1)^{N+1}\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)} \\
&\quad\times \left[1 + \sum_{n=1}^{N}(-1)^n \sum_{\ell_1<\ell_2<\cdots<\ell_n}\prod_{k=1}^{n}\big(x_{\ell_k} - x_i\big)\big(y_{\rho(\ell_k)} - y_j\big)\right]
\end{aligned}
$$

$$= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho\in S_N} \epsilon(\rho) \frac{(-1)^{N+1}\,\mathrm{e}^{\sum x_\ell y_{\rho(\ell)}}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}$$

$$\times \left[1 + \sum_{n=1}^{N}(-1)^n \sum_{\substack{\ell_1<\ell_2<\cdots<\ell_n\\ \ell_k\neq i,\rho(\ell_k)\neq j,\forall k}} \prod_{k=1}^{n}\left(x_{\ell_k} - x_i\right)\left(y_{\rho(\ell_k)} - y_j\right)\right]. \qquad \text{(A.1)}$$

Let us consider the following subexpression from the above formula

$$\frac{(-1)^{N+1}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}\left[1 + \sum_{n=1}^{N}(-1)^n \sum_{\substack{\ell_1<\ell_2<\cdots<\ell_n\\ \ell_k\neq i,\rho(\ell_k)\neq j,\forall k}} \prod_{k=1}^{n}\left(x_{\ell_k} - x_i\right)\left(y_{\rho(\ell_k)} - y_j\right)\right]$$

$$\stackrel{(\star)}{=} (-1)^{N+1}\left[\frac{1}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}\right.$$

$$\left. + \sum_{n=1}^{N-1}(-1)^n \sum_{\substack{\ell_1<\ell_2<\cdots<\ell_n\\ \ell_k\neq i,\rho(\ell_k)\neq j,\forall k}} \frac{\prod_{k=1}^{n}\left(x_{\ell_k} - x_i\right)\left(y_{\rho(\ell_k)} - y_j\right)}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_{\rho(\ell)} - y_j)}\right]$$

$$= (-1)^{N+1}\left[\frac{1}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}\right.$$

$$\left. + \sum_{n=1}^{N-1}(-1)^n \sum_{\substack{\ell_1<\ell_2<\cdots<\ell_n\\ \ell_k\neq i,\rho(\ell_k)\neq j,\forall k}} \frac{1}{\prod_{\ell\notin\{i,\ell_k,\forall k\}}(x_\ell - x_i)\prod_{\rho(\ell)\notin\{j,\rho(\ell_k),\forall k\}}(y_{\rho(\ell)} - y_j)}\right]$$

$$= \left[\frac{(-1)^{N+1}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}\right.$$

$$\left. + \sum_{n=1}^{N-1}(-1)^{N-n+1} \sum_{\substack{i_1<\cdots<i_{N-n}:\\ i\in\{i_k\},\, j\in\{\rho(i_k)\}}} \frac{1}{\prod_{k:i_k\neq i}\left(x_{i_k} - x_i\right)\prod_{k:\rho(i_k)\neq j}\left(y_{\rho(i_k)} - y_j\right)}\right]$$

$$= \left[\frac{(-1)^{N+1}}{\prod_{\ell\neq i}(x_\ell - x_i)\prod_{\ell\neq j}(y_\ell - y_j)}\right.$$

$$\left. + \sum_{n=1}^{N-1}(-1)^{n+1} \sum_{\substack{i_1<\cdots<i_n:\\ i\in\{i_k\},\, j\in\{\rho(i_k)\}}} \frac{1}{\prod_{k:i_k\neq i}\left(x_{i_k} - x_i\right)\prod_{k:\rho(i_k)\neq j}\left(y_{\rho(i_k)} - y_j\right)}\right]$$

$$= \left[\sum_{n=1}^{N} (-1)^{n+1} \sum_{\substack{i_1<\cdots<i_{N-n}:\\ i\in\{i_k\},\, j\in\{\rho(i_k)\}}} \frac{1}{\prod_{k:i_k\neq i}\left(x_{i_k}-x_i\right)\prod_{k:\rho(i_k)\neq j}\left(y_{\rho(i_k)}-y_j\right)} \right]$$

$$\stackrel{(\star\star)}{=} \left[\sum_{n=1}^{N}(-1)^{n+1}\sum_{i_1<\cdots<i_n} \right.$$

$$\left. \times \frac{\sum_{s=1}^{n}(-1)^s\delta_{i,i_s}\Delta\left(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots x_{i_n}\right)\sum_{s=1}^{n}(-1)^s\delta_{j,\rho(i_s)}\Delta\left(y_{\rho(i_1)},\ldots,\widehat{y_{\rho(i_s)}},\ldots y_{\rho(i_n)}\right)}{\Delta\left(x_{i_1},\ldots,x_{i_n}\right)\Delta\left(y_{\rho(i_1)},\ldots,y_{\rho(i_n)}\right)} \right]$$

$$= \sum_{n=1}^{N}(-1)^{n+1}\sum_{i_1<\cdots<i_n} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ \delta_{ii_1} & \cdots & \delta_{ii_{n+1}} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n} & \cdots & x_{i_{n+1}}^{n} \end{pmatrix}} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n-1} & \cdots & y_{\rho(i_{n+1})}^{n-1} \\ \delta_{j\rho(i_1)} & \cdots & \delta_{j\rho(i_{n+1})} \end{pmatrix}}{\det\begin{pmatrix} 1 & \cdots & 1 \\ y_{\rho(i_1)} & \cdots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n} & \cdots & y_{\rho(i_{n+1})}^{n} \end{pmatrix}}. \qquad \text{(A.2)}$$

In the above chain of equality we have replaced (after the ($\star$)) the upper limit of summation by $N-1$ because in the case $n = N$ there is certainly one ℓ_k equal to i (and one $\rho(\ell_k)$ equal to j) so that that term does not contribute. After ($\star\star$) we have removed the condition on the multi-index because it is implicit in the sum of deltas that follows. Putting together equation (A.2) with equation (A.1) or (2.8) we obtain the desired proof of the equivalence of formula (2.4) with equation (2.5), thus also proving the former rigorously, which was not done in [16].

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